Existence and Approximation of Traveling Wavefronts of a Diffusive Hematopoiesis Model with Time Delay

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1. Introduction

Recently there have been intensive studies on the existence of traveling waves to delayed reaction-diffusion equations arising from biological and physical applications, see for example Murray [1]. For delayed nonlinear reaction-diffusion equations with quasimonotonicity, nonlinearity, monotone-convergence scheme coupled with the standard upper-lower solution technique, developed by J. Wu and X. Zou [2], is widely used to establish the existence of monotone waves. See [3, 5].

In this work, traveling wavefront solutions including waves that move at minimum speed, will be investigated for the partial differential equation:

$$u_t(t,x)=\Delta u(t,x)+\frac{pu(t-\tau,x)}{1+au^q(t-\tau,x)}-du(t,x), \qquad (1)$$

where $x\in\mathbb{R}$, $t>0$, $u\geq 0$, and all parameters are positive constants. This equation, without spatial dispersion, was first suggested in 1977 by Mackey and Glass [6], among others, to model the concentration of cells in the circulating blood and where τ is the time delay between the production of immature stem cells in bone marrow and their maturation for release in the circulating blood stream.

We develop an alternative proof of the existence of such solutions, including the existence of traveling wavefronts moving at minimum speed. The proposed approach is based on the use technique of upper-lower solutions. Finally, through the iterative procedure developed in [2], we show numerical simulations that approximate the traveling wavefronts, thus confirming our theoretical results.

2. Preliminaries

A traveling wavefront solution to equation (1) is a special type of bounded positive continuous non-constant solution $u(t,x)$ having the form $u(t,x)=\phi(x+ct)$. The number $c>0$ is called the wave speed of the propagation, and $\phi:\mathbb{R}\to\mathbb{R}$ is a C^2-smooth function called the wave profile and satisfying $\phi(-\infty)=0$, $\phi(+\infty)=k>0$. The existence of the traveling wavefronts in (1) is equivalent to the presence of positive heteroclinic connections in an associated second order non-linear differential equation:

$$L_\phi=\phi''(z)-c\phi'(z)+\frac{p\phi(z-cr)}{1+a[\phi(z-cr)]^q}-d\phi(z)=0, \quad z\in\mathbb{R}. \qquad (2)$$

We look for traveling wavefronts solutions of (1) in the following profile set:

$$\Gamma=\left\{\begin{array}{l}\phi\in C(\mathbb{R},\mathbb{R}):\phi \textit{ is nondecreasing and}\\ \qquad\phi(-\infty=0),\ \phi(+\infty=k)\end{array}\right.$$

.
A continuous function $\rho\in C(\mathbb{R},\mathbb{R})$ is called an upper solution of (2) if ρ' and ρ'' exist almost everywhere in $\mathbb{R}$, and they are essentially bounded on $\mathbb{R}$ and if the following inequality holds:

$$\rho''(z)-c\rho'(z)+f_c(\rho_z)\leq 0, \quad z\in\mathbb{R}. \qquad (3)$$

A lower solution of (2) is defined in a similar way by reversing the inequality in (3). Where

$$f_c(\phi_z)=\frac{p\phi_z(-cr)}{1+a[\phi_z(-cr)]^q}-d\phi_z(0), \quad \phi_z(s)=\phi(z+s).$$

Theorem 1 *Assume that the following conditions on f holds.*

- **(F1)** *There Exists $k>0$ such that $f_c(\hat{0})=f_c(\hat{k})=0$, and $f_c(\hat{u})\neq 0$ for $0<u<k$, where $\hat{u}$ denotes the constant function taking the values u on $[-c\tau,0]$, i.e., $\hat{u}(s)=u$, $s\in[-c\tau,0]$.*
- **(F2)** *(Quasimonotonicity). There Exists $\beta\geq 0$ such that $f_c(\phi_z)-f_c(\psi_z)+\beta[\phi_z(0)-\psi_z(0)]\geq 0$, with $\phi_z,\psi_z\in X_c$ and $0\leq\psi_z(s)\leq\phi_z(s)\leq k$, $s\in[-c\tau,0]$.*

Suppose that (2) has an upper solution $\bar{\phi}\in\Gamma$ and a lower solution $\underline{\phi}$ (which is not necessarily in Γ) satisfying:

- **(S1)** $\underline{\phi}(z)\not\equiv 0, \quad z\in\mathbb{R}$
- **(S2)** $0<\underline{\phi}(z)\leq\bar{\phi}(z)\leq k, \quad z\in\mathbb{R}.$

Then, (2) have a solution in Γ . That is, (1) has a traveling wave front with speed c.

3. Results

Theorem 2 *There exists $c_*>0$ such that for every $c\geq c_*$, the equation (1) has a positive monotone traveling wavefront $u(t,x)=\phi(x+ct)$, connecting 0 with $k=\left(\frac{p-d}{ad}\right)^{1/q}$, if one of the following conditions holds:*

(a) $1<\dfrac{p}{d}\leq\dfrac{q}{q-1}$ *if* $q>1$*;*

(b) $1<\dfrac{p}{d}\leq+\infty$ *if* $0<q\leq 1$.

Lemma 1. If $\frac{p}{d}>1$, then there exists $k>0$ such that $f_c(\hat{0})=f_c(\hat{k})=0$, and $f_c(\hat{u})\neq 0$ for $0<u<k$, where $\hat{u}(s)=u$, $s\in[-c\tau,0]$.

Proof. Clearly $k=\left(\frac{p-d}{ad}\right)^{1/q}>0$, since $\frac{p}{d}>1$. So by computing the stationary states of (1), we get the result.

Lemma 2 If (a) or (b) holds, then for all $\beta\geq d$, f_c satisfies the quasimonotonicity condition.
Proof Let $\phi_z,\psi_z\in X_c$ be, such that $0\leq\psi_z(s)\leq\phi_z(s)\leq k$, $s\in[-c\tau,0]$. Then

$$f_c(\phi_z)-f_c(\psi_z)=p\left(\frac{\phi_z(-c\tau)}{1+a[\phi_z(-c\tau)]^q}-\frac{\psi_z(-c\tau)}{1+a[\psi_z(-c\tau)]^q}\right)$$

-d$(\phi_z(0)-\psi_z(0))$.
We consider the function $g(y)=\dfrac{py}{1+ay^q}$. We notice that $g'(y)=\frac{p[1+ay^q(1-q)]}{(1+ay^q)^2}\geq 0$, for all $y\in\mathbb{R}$, since $0<q\leq 1$, in the case that $q>1$, $g'(y)\geq 0$, for all $y\in\left[0,\frac{1}{(aq-a)^{1/q}}\right]$ and $0<k\leq\left(\frac{1}{(aq-a)^{1/q}}\right)$, because $\frac{p}{d}\leq\frac{q}{q-1}$. So the function g is non-decreasig. Therefore

$$f_c(\phi_z)-f_c(\psi_z)+d[\phi(z)-\psi(z)]\geq 0.$$

Existence of traveling wavefront solution: Case $c>c_*$

$$\bar{\phi}(z)=\begin{cases}\dfrac{k(\eta-\mu_1)}{\lambda_1-\mu_1+\eta}e^{z\lambda_1} & \text{si } z<0\\ k-\dfrac{\lambda_1 k}{\lambda_1-\mu_1+\eta}e^{z(\mu_1-\eta)} & \text{si } z\geq 0,\end{cases} \qquad (4)$$

where $\eta\geq\eta_0=\dfrac{2\mu_1-c+\sqrt{(c-2\mu_1)^2+4p_1e^{-\mu_1 c\tau}}}{2}$, is upper solution of (2).

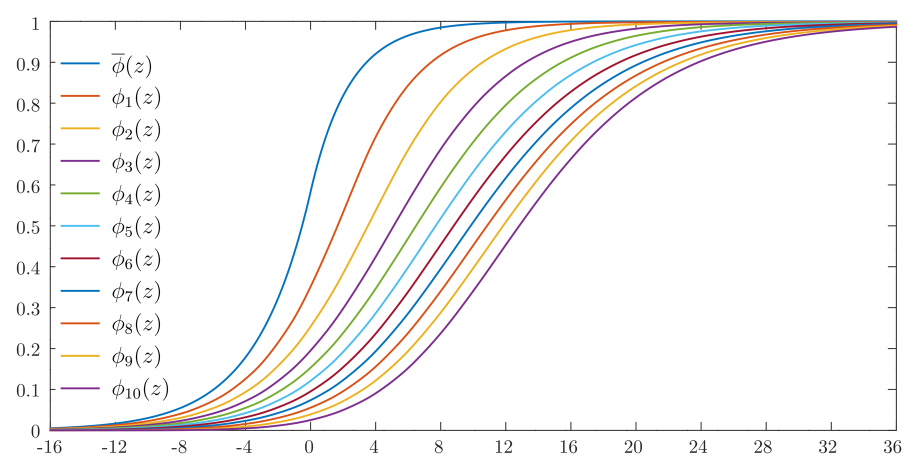

Figura 1: *First 10 iterations for $c=2$ and $q=1$ $a=e-1$, $\tau=1$, $p=e$ and $d=1$*

Existence of traveling wavefront solution: Case $c=c_*$

$$\bar{\phi}_*(z)=\begin{cases}\dfrac{k(\eta_*-\mu_1)}{\lambda_*+2(\eta_*-\mu_1)}(2-\lambda_* z)e^{z\lambda_*} & \text{si } z<0\\ k-\dfrac{\lambda_* k}{\lambda_*+2(\eta_*-\mu_1)}e^{z(\mu_1-\eta_*)} & \text{si } z\geq 0,\end{cases} \qquad (5)$$

where $\eta_*\geq\eta_0$, is upper solution of (2).

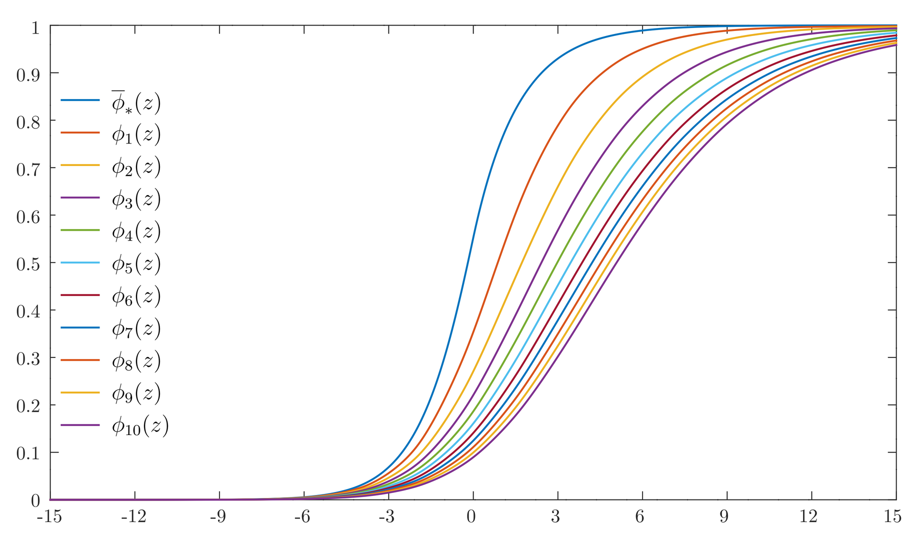

Figura 2: *First 10 iterations for $q=1$ $a=e-1$, $\tau=1$, $p=e$ and $d=1$ and thus $c_*(\tau,p,d)=1$*

4. Acknowledgements

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