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Bovine Babesiosis model with feedback control variables

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Resumen

We propose and study a continuous model with feedback controls for Babesiosis disease in bovine and tick populations. The global stability of the boundary-equilibrium point of this model is analyzed by means of rigorous mathematical methods. As an important consequence of this result, we formulate a strategy to select feedback control variables in order to restrain the disease in the original model. This strategy allows us to make the disease vanish completely. In other words, the feedback controls are especially effective for restraining disease in the model. The validity of the established theoretical result is supported by a set of numerical simulations.

1. Introduction

Bovine babesiosis is the most important arthropod-borne disease of cattle worldwide. It provokes morbidity and even mortality of cattle after a tick-borne, parasitic infection. Ticks infected due to the ingestion of parasites in the blood of infected cattle are the most relevant transmission agent of such a disease. The permanence of the infection depends on the probability of the vertical transmission in ticks population. Control strategies based on vaccination and antiparasitic treatments have been performed. But, due to residues and other problems, some vaccines and drugs have been eliminated from these strategies (see [4] and references therein).
In reality, we always desire that the disease in the original model will be eliminated or restrained. This is an important objective but not simple in real-world applications. The complete global stability of the model suggests that we should choose the parameters of the model such that the reproduction number is less than or equal 1. This brings up a good plan to protect the population from the disease. However, when this cannot be done, i.e., when the reproduction number is greater 1, then we must accept that there always exist infected individuals within the population. In this case, we have to propose the strategies in order to eliminate or restrain the disease. For this purpose, we can consider the model with feedback controls as an efficient approach. For the models with feedback controls, we refer the readers to [2, 5] and references therein. Up to now, many results on ecosystem with

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feedback controls were proposed and analyzed with many different objectives. Below we mention some notable models in this topic.
Up to present, the Babesiosis disease model with feedback controls has not been studied yet. This motivates us to introduce feedback controls into the model and investigate its dynamics with the aim to use these feedback controls to restrain or eliminate the Babesiosis disease. Namely, in this paper, following [2, 5], we introduce three feedback control variables to all three components of the model. Using mathematical analysis, we obtain the result that, by suitable selection of feedback controls, we can completely restrict the disease.

2. Mathematical model

The dynamic transmission of Babesiosis disease for bovine and tick populations can be modeled by the following system of nonlinear first order differential equations

$$\begin{cases} \bar{S}'_B(t) = \mu_B(\bar{S}_B(t) + \bar{C}_B) + \alpha_B\bar{C}_B(t) - \mu_B\bar{S}_B(t) - \beta_B\bar{S}_B(t)\dfrac{\bar{I}_T(t)}{N_T(t)}, \\ \bar{I}'_B(t) = \mu_B\bar{I}_B + \beta_B\bar{S}_B(t)\dfrac{\bar{I}_T(t)}{N_T(t)} - \mu_B\bar{I}_B - \lambda_B\bar{I}_B(t), \\ \bar{C}'_B(t) = \lambda_B\bar{I}_B(t) - (\mu_B + \alpha_B)\bar{C}_B(t), \\ \bar{S}'_T(t) = \mu_T(\bar{S}_T + p\bar{I}_T) - \beta_T\bar{S}_T(t)\dfrac{\bar{I}_B(t)}{N_B(t)} - \mu_T\bar{S}_T(t), \\ \bar{I}'_T(t) = \beta_T\bar{S}_T(t)\dfrac{\bar{I}_B(t)}{N_B(t)} + (1-p)\mu_T\bar{I}_T(t) - \mu_T\bar{I}_T(t), \end{cases} \tag{1}$$

where the total population of bovine $N_B(t)$ is divided into three subpopulations: bovines which may become infected (Susceptible $\bar{S}_B(t)$); bovines infected by the Babesia parasite (Infected $\bar{I}_B(t)$); and bovines which have been treated for the Babesiosis (Controlled $\bar{C}_B(t)$). The total population of ticks $N_T(t)$ is divided into two subpopulations: Ticks which may become infected $\bar{S}_T(t)$ and ticks infected by the Babesia parasite $\bar{I}_T(t)$. To be biologically significant, all parameters in this model are positive. Details on this model can be found in [1].

Mathematical analyzes show that system (1) is equivalent to the following subsystem:

$$\begin{cases} S'_B(t) = (\mu_B + \alpha_B)(1 - S_B(t) - I_B(t)) - \beta_B S_B(t) I_T(t), \\ I'_B(t) = \beta_B S_B(t) I_T(t) - \lambda_B I_B(t), \\ I'_T(t) = \beta_T(1 - I_T(t)) I_B(t) - \mu_T p I_T(t), \end{cases} \tag{2}$$

where, the

$$\Omega = \{(S_B, I_B, I_t) | 0 \leq S_B + I_B \leq 1, 0 \leq I_T \leq 1\} \tag{3}$$

is a positive invariant set for the system (2). For system (2), it is adopted the notation (see [1])

$$\mathcal{R}_0 = \frac{\beta_B\beta_T}{\lambda_B\mu_T p} \tag{4}$$

Concerning system (2) the following results are obtained in [1]:

(i) System (2) has the disease free equilibrium $F_1^* = (S_{B_1}^*, I_{B_1}^*, I_{T_1}^*) = (1, 0, 0)$ for all values of the parameters in this system, whereas, only if $\mathcal{R}_0 > 1$, there is a (unique) endemic equilibrium $F_2^* = (S_{B_2}^*, I_{B_2}^*, I_{T_2}^*)$ in the interior of Ω given by:

$$S_{B_2}^* = \frac{\beta_T\lambda_B(\alpha_B + \mu_B) + \lambda_B\mu_T p(\alpha_B + \lambda_B + \mu_B)}{\beta_T\alpha_B(\beta_B + \lambda_B) + \beta_T\lambda_B\mu_B + \beta_T\beta_B(\lambda_B + \mu_B)},$$

$$I^*_{B_2} = \frac{(\mu_B + \alpha_B)(\beta_B\beta_T - \lambda_B\mu_T p)}{\beta_T\alpha_B(\beta_B + \lambda_B) + \mu_B\beta_T\lambda_B + \beta_T\beta_B(\lambda_B + \mu_B)},$$

and

$$I^*_{T_2} = \frac{(\mu_B + \alpha_B)(\beta_B\beta_T - \lambda_B\mu_T p)}{\beta_T\beta_B(\alpha_B + \mu_B) + \beta_B\mu_T p(\alpha_B + \lambda_B + \mu_B)}.$$

(ii) If $\mathcal{R}_0 \leq 1$, then the disease-free point F_1^* is globally asymptotically stable; otherwise, the disease-free point F_1^* is unstable.

(iii) If $\mathcal{R}_0 > 1$, then the endemic point F_2^* is shown to be locally asymptotically stable by numerical simulations.

It should be emphasized that the local stability of F_2^* had only been proved via numerical simulations [1]. In [3] we confirm this theoretically. Moreover, the global stability of the model also is established.

3. The model with feedback controls

The global stability of the model (2) suggests that we should choose the parameters such that $\mathcal{R}_0 \leq 1$. This brings up a good plan to protect the bovine population from the disease. However, when this cannot be done, i.e., $\mathcal{R}_0 > 1$, then we must accept that there always exist infected individuals within the population ($\lim_{t\to\infty} I_B(t) = I_B^* > 0$ and $\lim_{t\to\infty} I_T(t) = I_T^* > 0$). In this case, one of the most effective approaches to minimize the number of infected bovines and ticks is to consider the model with feedback controls.

Because of this motivation, inspired by the previous studies [2, 5] we consider system (2) with feedback controls as follows:

$$\begin{cases} S'_B(t) = (\mu_B + \alpha_B)(1 - S_B(t) - I_B(t)) - \beta_B S_B(t) I_T(t) - c_1 S_B(t) u_1(t), \\ I'_B(t) = \beta_B S_B(t) I_T(t) - \lambda_B I_B(t) - c_2 I_B(t) u_2(t), \\ I'_T(t) = \beta_T(1 - I_T(t)) I_B(t) - \mu_T p I_T(t) - c_3 I_T(t) u_3(t), \\ u'_1(t) = d_1 S_B(t) - e_1 u_1(t), \\ u'_2(t) = d_2 I_B(t) - e_2 u_2(t), \\ u'_3(t) = d_3 I_T(t) - e_3 u_3(t), \end{cases} \tag{5}$$

where u_1, u_2 and u_3 are feedback control variables and the parameters c_i, di and e_i $(i = 1, 2, 3)$ are positive constants. The initial conditions are $S_B(0) > 0$, $I_B(0) > 0$, $I_T(0) > 0$, $u_i(0) > 0$. The vaccination term here is not a vaccinated population in the context of the controlled epidemic model. It is a feedback vaccination control driven by the three subpopulation and with decreasing transient terms due to the initial condition of the vaccination. This is clearly seen by integrating through time in the three last equation of (5). Our main objective is to prove that the feedback control variables can be used for restraining the number of infected bovines and ticks. For this purpose, we need to investigate the global stability of model (5).

3.1. A Lyapunov function to analyze the global stability of model with feedback controls

It is easy to verify that the set $\mathbb{R}^6_+$ is a positive invariant set of model (5).

Theorem 3.1 *(The boundary-equilibrium point and the reproduction mumber). For the model (5), we have:*

(i) *It always possesses a unique boundary-equilibrium point*

$$E_0 = \left(S_B^0, I_B^0, I_T^0, u_1^0, u_2^0, u_3^0\right) = \left(S_B^0, 0, 0, \frac{c_1 d_1}{e_1} S_B^0, 0, 0\right)$$

for all values of the parameters, where

$$S_B^0 = \frac{-(\mu_B + \alpha_B) + \sqrt{(\mu_B + \alpha_B)^2 + 4(\mu_B + \alpha_B)\frac{c_1 d_1}{e_1}}}{2\frac{c_1 d_1}{e_1}}. \tag{6}$$

(ii) *The basic reproduction number of the model is*

$$\mathcal{R}_0^{new} := \frac{\beta_B \beta_T}{\lambda_B \mu_T p} S_B^0, \tag{7}$$

where S_B^0 is given by (6).

See [3] for proof of theorem.

Theorem 3.2 *The boundary-equilibrium point E_0 is globally asymtotically stable of model (5) if the reproduction number $\mathcal{R}_0^{new} \leq 1$.*

See the prove in [3].

3.2. A strategy of the selection of the feedback control variables

Based on Theorem 3.2, we propose a strategy to restrain the disease in model (5) in the case $\mathcal{R}_0 > 1$. This strategy is based on the selection of the feedback control variables such that $\mathcal{R}_0^{new} \leq 1$.

Consider model (2) in case of $\mathcal{R}_0 > 1$. In this case, the unique positive equilibrium point F_2^* is globally asymptotically stable. Consequently, $\lim_{t\to\infty} I_B(t) = I_B^*$ and $\lim_{t\to\infty} I_T(t) = I_T^*$. It is worthy to note that the reproduction number $\mathcal{R}_0^{new}$ of model (5) depends not only on the parameters of the original differential model (2) but also on the feedback control variables. Suppose that the feedback control variables are selected such that $\mathcal{R}_0^{new} \leq 1$, then the boundary-equilibrium point of model (5) becomes globally asymptotically stable. As an important consequence of this result, we have

$$\lim_{t\to\infty} I_B(t) = 0, \quad \lim_{t\to\infty} I_T(t) = 0,$$

which means that the disease is restricted completely.
Now, we prove the following proposition of how to select control variables so that $\mathcal{R}_0^{new} \leq 1$.

Proposition 3.3 *For the reproduction number $\mathcal{R}_0^{new}$ of the model (5), we have $\mathcal{R}_0^{new} \leq 1$ if and only if the following condition holds*

$$\frac{c_1 d_1}{e_1} \geq (\mu_B + \alpha_B)\mathcal{R}_0(\mathcal{R}_0 - 1). \tag{8}$$

where $\mathcal{R}_0$ is defined by (4) for the model (2).

See prove in [3].

Therefore, form the above proposition 3.3 and Theorem 3.2, it follows that we can choose the parameters c_1, d_1 and e_1 to ensure that the disease will vanish. The following scenario is presented to confirm the strategy.

Consider system (2) with the parameters (see [1, 4]). In this case $\mathcal{R}_0 = 67.5 > 1$ and the equilibrium $F_2 = (0.0497, 0.7893, 0.7019)$ is globally asymptotically stable. Therefore, $\lim_{t\to\infty} I_B(t) = 0.7983 > 0$, and $\lim_{t\to\infty} I_T(t) = 0.7019 > 0$. We select the feedback control variables in model (5) as follows $c_1 = 2$, $c_2 = 3$, $c_3 = 2$, $d_1 = 4$, $d_2 = 1$, $d_3 = 1$, $e_1 = 1$, $e_2 = 1$, $e_3 = 2$.
In this case, $\mathcal{R}_0 = 0.8578 < 1$. By Theorem 3.2, we conclude that the boundary-equilibrium point $E_0 = (0.0127, 0, 0, 0.0507, 0, 0)$ of model (5) is globally asymptotically stable. This implies that $\lim_{t\to\infty} I_B(t)^{new} = \lim_{t\to\infty} I_T(t)^{new} = 0$. The three first components S_B, I_B, I_T of some solutions to the model (5) are depicted in the panel b Figure 1. From this figure, it is clear the feedback control variables allow us to make the disease in the original model vanish completely.

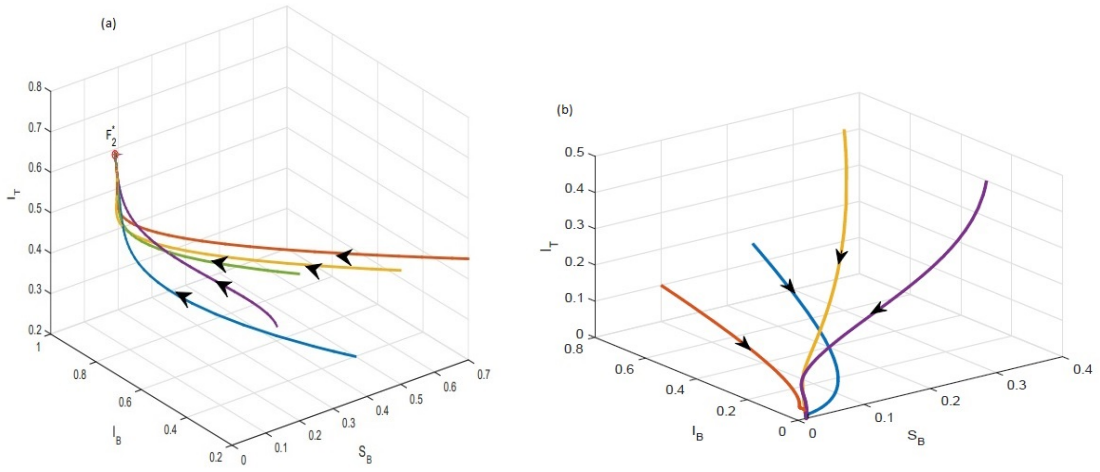

Figura 1: Parameter values: $\mu_B = 0.0002999$, $\mu_T = 0.001609$, $\beta_B = 0{,}00061$, $\beta_T = 0.00048$, $\lambda_B = 0{,}000265$, $p = 0{,}1$. Panel (a) $\mathcal{R}_0 = 67.544$ and $F_2 = (0.3722, 0.5215, 0.6087)$ for model 2. Panel (b) $\mathcal{R}_0^{new} = 0.8578$ and $E_0 = (0.0127, 0, 0, 0.0507, 0, 0)$ for model (5).

For model (5), we can predict that if $\mathcal{R}_0^{new} > 1$, then this model possesses the unique positive equilibrium point and it is globally asymptotically stable. This prediction is an interesting problem from a theoretical standpoint, but it is not simple because the model (5) has a large dimension and contains many parameters. We can easily confirm the prediction by numerical simulations. However, in the proposed strategy, we do not need to care about the stability properties of the unique positive equilibrium point.

4. Conclusions and future research directions

The main contribution of the paper is that we introduce feedback controls into the original model and establish, both theoretically and experimentally, the global asymptotical stability of a unique boundary equilibrium when the reproduction number of the extended model with feedback controls is not greater than one. Because of the selection of appropriate control variables, we can make the system to approach to the unique boundary equilibrium, which corresponds to the disease-free equilibrium. This provides an effective method for restraining the disease. From the biological

point of view, we can understand the feedback control variables as a vaccination driven by the three subpopulations. An interesting case is to study the case when this vaccination is driven only by the population of susceptible bovine. As future research directions, we shall try to use the techniques employed in this work to other applied models in order to get results on the global stability of such models when implementing feedback controls. The construction of discrete models preserving essential properties of these continuous models will be also subject of our research.

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