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Influence of linear transmission rate on the asymptotic behavior of a stochastic epidemic model.

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Resumen

Based on the model of a recently published work [3], we seek to know the development of the process over time but in this instance under a linear incidence rate of the disease. The limit of the deterministic model is obtained based on the parameters of the model. The stochastic model is posed when environmental variability is included in the transmission rate which leads to a stochastic differential equation ([2]), which is analyzed. Conditions are found for the extinction of the disease that occur along with numerical simulations of the process. [11]

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1. Introduction

It is well known that various kinds of infectious diseases play important roles in the world not only threatening people's lives but also due to catastrophic economic consequences produced in

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countries. In order to suppress the outbreak of such infectious diseases and reduce the loss brought by those epidemics, many deterministic compartmental models have been proposed and investigated, which are expressed by ordinary differential equations (ODE).

Nevertheless, there are many factors which would produce some uncertain effects on these models and stochasticity can be incorporated on models to study the behavior of certain infectious diseases usually taking stochastic disturbances into account in the deterministic model, in the form of Gaussian white noises.

In this paper, random factors are introduced and a model described by a stochastic differential equation (SDE) is studied, considering a fatal disease in an animal population, similar to the model proposed by Roberts and Saha [13] to describe the dynamics of bovine tuberculosis in possum populations in New Zealand [6], a zoonotic disease caused by a bacteria called Mycobacterium bovis (M.bovis), which is closely related to the bacteria that causes human and avian tuberculosis [12]. The model proposed by [13] can also be used to study other epidemics, such as AIDS transmission [4, 5].

The stochastic model incorporating some features of asymptotic behavior of a disease is based on a deterministic model with an added term to account for the effect of environmental fluctuations on the disease transmission rate and pseudo-vertical transmission as assumed in [13].

In order to describe epidemic more reasonably, we suppose the disease transmission rate is described by a linear incidence rate. We think that this type of incidence rate is more flexible than that proposed by Roberts and Saha [13], since it allows changes in the transmission rate according to the size of the infected proportion.

2. Description of the model

In order to describe the dynamics of epidemics, Roberts and Saha consider a population of density N individuals per unit of area, with natural density-dependent birth and death rates given by $B(N)$ and $D(N)$, respectively. Assuming animals may be classified as either susceptible to infection (density S) or infected and infectious (density I), then $N = S + I$. Designing by $Z(t) = \frac{I}{N}$ the proportion of the host animals infected with disease against the total population, then the model is described by

$$X_\mu : \begin{cases} \frac{dN}{dt} &= (B(N) - D(N) - \alpha Z)\, N \\ \frac{dZ}{dt} &= -(1-p)\, B(N)\, Z + (\beta C(N) - \alpha)(1-Z)\, Z \end{cases} \tag{1}$$

where $\mu = (\alpha, \beta, p) \in \mathbb{R}^3_+$. Moreover, $\alpha \geq 0$ is the increase of the death rate resulting from the infectious disease [6], p is the vertical distribution probability (mother to offspring) of the disease, i.e. $0 \leq p \leq 1$, $C(N)$ is the contact rate between individuals in the population and β is the disease transmission rate [12].

After some analytical considerations [12, 13] assuming $\frac{dN}{dt}$ tends to 0 a lot faster than $\frac{dZ}{dt}$ then it can be taken $B(N) = B$ and $C(N) = C$, and the following equation is obtained instead of the model (1)

$$\frac{dZ}{dt} = -(1-p)\, BZ + (\beta C - \alpha)(1-Z)\, Z, \tag{2}$$

with $p - 1 \leq 0$, $B > 0$ and $C > 0$. Notice that β, the disease transmission rate, is constant for this model, i.e., it is not dependent on the value of the proportion of the population infected. It is easy to see that (2) is equivalent to the well-known logistic equation with intrinsic growth rate [6, 10].

Roberts and Saha [13], and Liu et al. [12] find the solution to this model and the limit of $Z(t)$ when t tends to ∞.

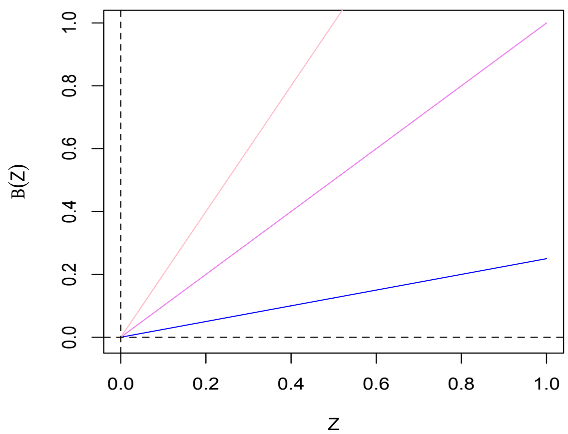

Figura 1: The Figure shows three alternatives for Beta (Z), a = 0.25 (blue line), a = 1 (violet line), a = 2 (pink line). The transmission rate will grow slow, regular or fast, respectively, in function of the proportion of infected population.

They also study the asymptotic behaviour after adding an additional component to the model due to environmental disturbance. In fact, the constant β by $\beta_0 + \rho\eta(t)$ is changed where rho is the enviromental disturbance intensity and $\eta(t)$ is a white noise. Both obtain the probability distribution $P^*(Z)$ of the proportion of population infected when Z reaches its steady state. Roberts and Saha [13] approach the mean and variance of this probability distribution and Liu et al. [12] study some properties of $P^*(Z)$ (modes, etc). In this work we present a similar approach applied to model (2) but where the disease transmission rate is no longer constant but dependent on the proportion of infected population.

Based on the model proposed in [13], an extension is made considering the disease transmission rate dependent on the proportion of population infected, i.e., we consider a linear incidence rate β, given by

$$\beta\,(Z) = aZ$$

for $0 < Z \leq 1$ and additional parameter $a > 0$, obtaining the model equation,

$$\frac{dZ}{dt} = -\,(1-p)\,BZ + ((aZ)C - \alpha)\,(1-Z)\,Z \tag{3}$$

We will consider this transmission rate β for populations where the proportion of infected population, Z, is positive then we take $Z \in [Z_0, 1]$, with $0 < Z_0 < 1$.

The rate $\beta(Z)$ changes fast or slow according to the value of the parameter a, depending on $a > 1$ or $a < 1$ as it is shown in Figure 1.

3. The deterministic model and the limit of $Z(t)$

In the following theorem we impose conditions over the parameters for which the proportion of infected population can be non-extinguished. Hereafter will be denoted by J by

$$J = (1-p)B + \alpha. \tag{4}$$

Let

$$p\,(Z) = -aCZ^2 + (aC+\alpha)\,Z - ((1-p)B+\alpha) \tag{5}$$

and the discriminant

$$\triangle := (aC+\alpha)^2 - 4aC\,((1-p)B+\alpha)\,. \tag{6}$$

Theorem

Assume that the polynomial $p(Z)$ described in (5) has real zeros $m \leq K$ and consider J defined in equation (4). Then the asymptotic behaviour of the proportion of infected population Z solution of the epidemic model given by (3) and initial condition $Z_0 \in (0,1)$ can be described by:

1. If $J = 0$ then $\alpha = 0$ y $p = 1$, therefore $m = 0$ y $K = 1$, then, when time passes all population is infected, that is,

$$\lim_{t\to\infty} Z(t) = 1.$$

2. If $J > 0$ y $\triangle > 0$ and $aC > \alpha$, then $0 < m < K \leq 1$ and:

$$\lim_{t\to\infty} Z(t) = \begin{cases} 0 & \text{si } Z_0 \leq m \\ K & \text{si } Z_0 > m. \end{cases} \tag{7}$$

   Moreover K = 1 if the offspring always inherits the disease from his mother, this is, if the probability of vertical transmission of the disease is 1. Therefore when K = 1 all population become infected when the time elapses, if the proportion of the population infected exceeds the m.

3. In other case, the disease is extinguished, that is,

$$\lim_{t\to\infty} Z(t) = 0.$$

4. Stochastic differential equation model

In this section we consider an stochastic model associated with the deterministic model in (3). The idea is to include in the model the environmental variability as in Roberts and Saha [13], Liu et al. [12] and Christen et al. [3]. In both cases, they assume that random factors affect the transmission of the disease. With this purpose they add a coefficient to the disease transmission rate. In our case, taking into account random perturbation to the disease transmission coefficient a in such a way that β becomes

$$\beta(Z) = (a_0 + \rho\eta(t))Z = (a_0 Z) + \rho\eta(t)Z$$

where $\eta(t)$ is a white noise and ρ is the environmental disturbance intensity. Then equation (3) becomes

$$dZ = F(Z)dt + G\,(Z)\,dW_t \tag{8}$$

where

$$F(Z) = -\,(1-p)\,BZ + (a_0 ZC - \alpha)\,(1-Z)\,Z, \tag{9}$$

$$G\,(Z) = \rho C Z^2 (1-Z). \tag{10}$$

and $(W_t)_{t\geq 0}$ is a standard Brownian motion. We are interested in proving that this SDE has a solution for all $t \geq t_0$, where t_0 is the initial time instant of the process, and that solution remains in the interval [0,1]. For that purpose we prove the following Lemmas.

Lemma
Consider the autonomous SDE

$$dZ = F(Z)dt + G\,(Z)\,dW_t, \quad t_0 \leq t < \infty$$

where $F(Z)$ and $G(Z)$ are defined by (9) and (10), respectively. Then there exists $K^* > 0$, $K^* = K^*(B, \alpha, a_0, C, \rho)$ such that the functions $F(Z)$ and $G(Z)$ satisfying the following global Lipschitz condition

$$|F(x) - F(y)| + |G(x) - G(y)| \leq K^* |x - y|$$

for any $x, y \in [0,1]$.

Lemma
The SDE considered in the previous Lemma subject to the initial condition $Z(t_0) = Z_0$, where Z_0 is a random variable such that $0 < Z_0 < 1$ and $E[(Z_0)^2] < \infty$, admits a unique global solution for all $t \in [t_0, \infty)$.

Lemma
The solution of the model (8) with initial condition $Z_0 \in (0,1)$ will remain in this interval for all $t > t_0$.

5. Extinction

Theorem
Let the stochastic epidemiological model (8), if: $R_0^s = \frac{a_0 C}{(1-p)B} < 1$ for $p < 1$, then for all initial condition $Z_0 \in (0,1)$, the process solution, Z_t, satisfies
$\limsup_{t\to\infty} \frac{1}{t} log(Z_t) \leq a_0 C - (1-p)B < 0,\ a.s.$
Thus, Z_t tends to 0 exponentially a.s, that is to say, the disease is extinguished with probability 1.

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